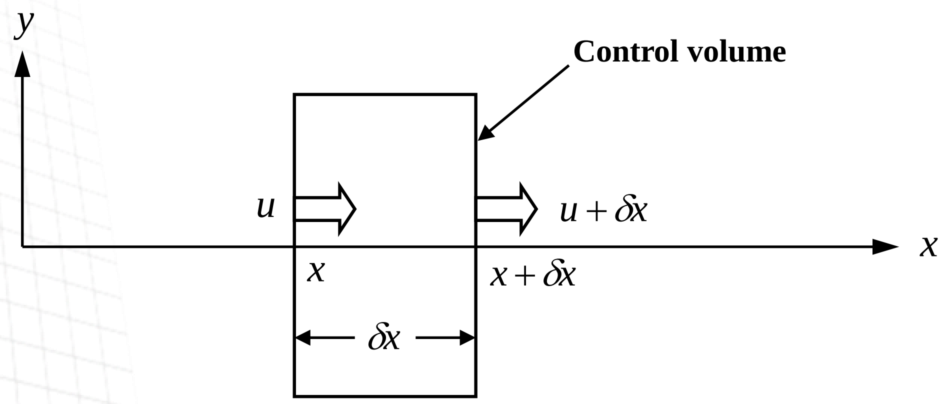
❖ The one-dimensional wave equation

- Flow variables
  - Pressure : $p = p(x,t)$
  - Density : $\rho = \rho(x,t)$
  - Velocity : $\vec{v} = \vec{v}(x,t)$

  (It is constrained to the x-direction) $\vec{v}(x,t) = \vec{v}(u(x),0,0,t)$
- Consider a control volume of unit length in the ***y*** and ***z*** directions and of width ***δx***.



The one-dimensional wave equation

- Since mass must be conserved, the change of mass within the control volume in unit time is related to the difference between the rate of inflow and outflow of the fluid crossing the end faces of element (Continuity)

$$\frac{\partial \rho'}{\partial t}\delta x = \{(\rho_0 + \rho')u\}(x,t) - \{(\rho_0 + \rho')u\}(x + \delta x, t)$$

- The small quantities can be negligible. Then we have Continuity equation.

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0$$

❖ The one-dimensional wave equation

- Similarly a linearized form of the equation of conservation of momentum, when all products of the small quantities have been neglected.

$$\rho_0 \frac{\partial u}{\partial t}\delta x = p'(x,t) - p'(x+\delta x, t)$$

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

- Differentiate the linearized continuity equation with respect to *t*, and subtract from the momentum equation differentiated with respect to the ***x***.

$$\frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0$$

Three-dimensional wave equation

❖ The one-dimensional wave equation

- In general an increase of pressure tends to increase the density and so the gradient is positive constant, ***c***.

$$p' = c^2 \rho'$$

- Now, wave equation is derived.

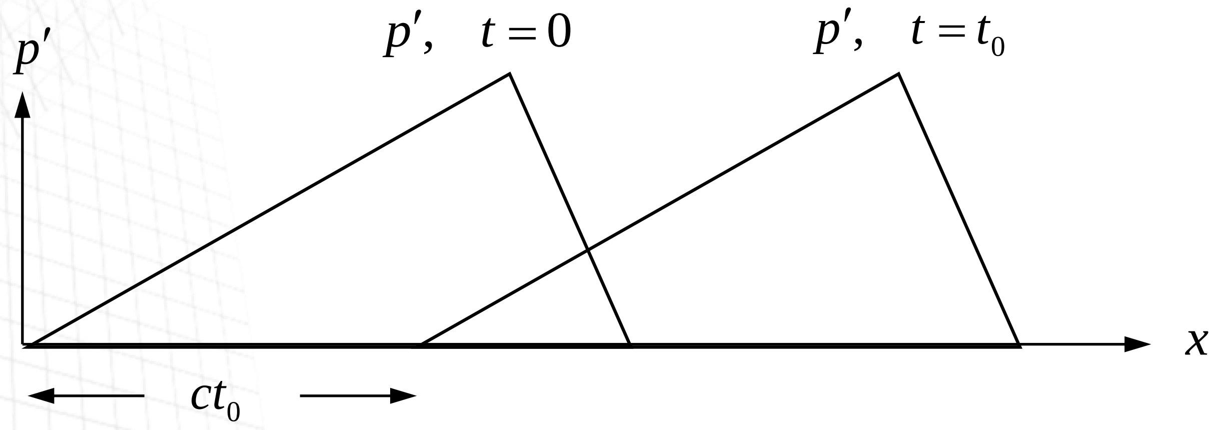
$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0$$

❖ The one-dimensional wave equation

- A general solution of the one-dimensional wave equation

$$p'(x,t) = f(x-ct) + g(x+ct)$$

- The pressure perturbation maintains the same form, but travels at speed *c*.



❖ The one-dimensional wave equation

Harmonic waves

- One particular form of waves is described by the solution of wave equation when f and g are harmonic functions

$$p'(x,t) = Ae^{i\omega(t-x/c)} + Be^{i\omega(t+x/c)}$$

where, ω is the frequency of the wave (rad./sec.)

2π/ω is the period of the wave

- In the plane wave, $p'(x,t) = Ae^{i\omega(t-x/c)}$

Then it becomes $p'(x,t) = \rho_0 c u'(x,t)$

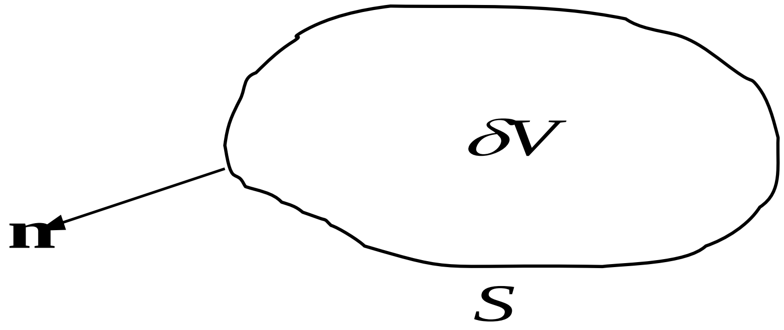
❖ The basic equation

● Conservation of mass

- Consider mass conservation for a small volume δV enclosed by surface S.

- The rate of increase of mass within the volume δV must be equal to the rate at which mass flows into the volume across the bounding surface

$$\delta V \frac{\partial \rho'}{\partial t} = -\int_S \rho_0 \mathbf{v} \cdot \mathbf{n} dS$$



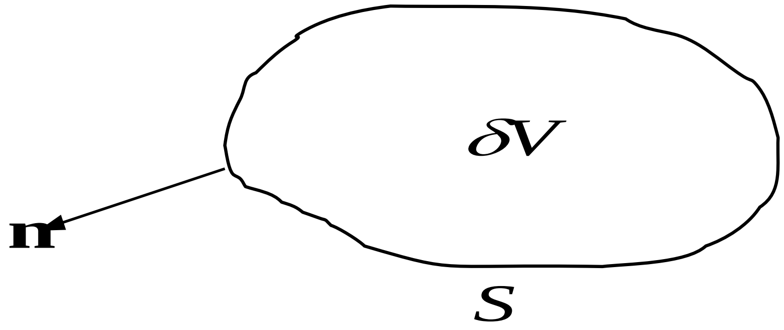
❖ The basic equation

- Conservation of mass
  - Gauss's theorem can be used to rewrite the surface integral, the linearized equation is derived.

$$\frac{\partial \rho'}{\partial t} + \rho_0 \mathrm{div}\,\mathbf{v} = 0$$

  - In tensor notation,

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v_i}{\partial x_i} = 0$$



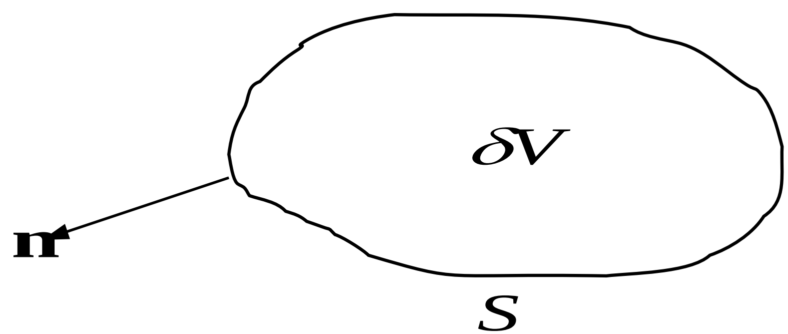
❖ The basic equation

- **Momentum equation**
  - The rate at which the momentum increases in unit volume of weakly disturbed matter must be equal to the force applied to that unit volume by neighboring material.

$$\rho_0 \frac{\partial v}{\partial t} + \text{grad } p' = 0$$

Force applied to unit volume by neighboring material

Momentum increased in unit volume of weakly disturbed matter



  - In tensor notation,

$$\rho_0 \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0$$

❖ The basic equation

- Wave equation
  - From the manipulation of mass conservation equation and momentum equation, wave equation is derived.

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v_i}{\partial x_i} = 0 \quad \times \quad \frac{\partial}{\partial t} \qquad \text{(Differentiate with respect to the t)}$$

$$- \left| \; \rho_0 \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0 \quad \times \quad \frac{\partial}{\partial x_i} \right. \qquad \text{(Divergence of equation)}$$

  - As a result, we can derive the wave equation.

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$$

❖ The basic equation

- Wave equation
  - (Proof) the velocity field satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial x_i}p^{'}\right)-\nabla^2\left(\frac{\partial}{\partial x_i}p^{'}\right)=0$$

$$\frac{\partial p^{'}}{\partial x_i}=-\rho_0\frac{\partial v_i}{\partial t}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\rho_0\frac{\partial v_i}{\partial t}\right)-\nabla^2\left(\rho_0\frac{\partial v_i}{\partial t}\right)=0$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}v_i^2-\nabla^2 v_i^2=0$$

❖ The basic equation

- The velocity potential
  - From momentum equation, taking curl both sides. Then we have

$$\rho_0 \frac{\partial}{\partial t} \nabla \times \vec{v} + \nabla \times \nabla p' = 0$$

  - Define vorticity : $\nabla \times \vec{v} \equiv \vec{\Omega}$

  - If the fluid is initially at rest, the flow is irrotational.

$$\frac{\partial}{\partial t} \vec{\Omega} = 0$$

  - The forces arising from the pressure gradient are conservative and act through the mass center of a particle. They can not induce spin or vorticity.

❖ The basic equation

- The velocity potential
  - Define velocity potential : $\vec{v} = \nabla \varphi$
  - Momentum equation becomes

$$\nabla \cdot \left(\rho_0 \left(\frac{\partial \varphi}{\partial t} \right) + p' \right) = 0$$

  - It means sum of pressure perturbation and $\rho_0 \dfrac{\partial \varphi}{\partial t}$ is a constant and can be set to zero.

$$p' = -\rho_0 \left(\frac{\partial \varphi}{\partial t} \right)$$

The basic equation

- The velocity potential
  - Using velocity potential, the wave equation becomes

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = 0$$

  - Laplace equation

(1) Incompressible flow $(c \to \infty)$

(2) Slow acoustic motion $\left(\frac{\partial^2}{\partial t^2} \to 0\right)$

(3) In the vicinity of a singularity $\left(\frac{\partial^2 \varphi}{\partial t^2} = 0\left(r^2 \nabla^2 \varphi\right)\right)$

  - The wave equation limits to Laplace's equation for 3 cases.

$$\nabla^2 \varphi = 0$$

The basic equation

- Differences between Aerodynamics and Acoustics
  - Steady Aerodynamics (Bernoulli equation)

$$p + \frac{1}{2}\rho u^2 = const.$$

  - Acoustics (Unsteady Bernoulli)

$$p' = -\rho_0 \left(\frac{\partial \varphi}{\partial t} \right)$$

❖ The basic equation

- The energetics of acoustic motion
  - Multiply the quantity $(\rho_0 / c^2)(\partial\varphi / \partial t)$ to the acoustic wave equation for the potential.

$$\frac{\rho_0}{c^2}\frac{\partial\varphi}{\partial t}\left\{\frac{\partial^2\varphi}{\partial t^2} - c^2\nabla^2\varphi = 0\right\}$$

  - The first term :

$$\frac{\partial\varphi}{\partial t}\frac{\rho_0}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial}{\partial t}\left\{\frac{1}{2}\frac{\rho_0}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2\right\} = \frac{\partial}{\partial t}\left\{\frac{p'^2}{2\rho_0 c^2}\right\}$$

  - The second term :

$$\frac{\rho_0}{c^2}\frac{\partial\varphi}{\partial t}c^2\nabla^2\varphi = \rho_0\frac{\partial\varphi}{\partial t}\frac{\partial^2\varphi}{\partial x_i\partial x_i} = \frac{\partial}{\partial x_i}\left\{\rho_0\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x_i}\right\} - \rho_0\frac{\partial^2\varphi}{\partial t\partial x_i}\frac{\partial\varphi}{\partial x_i}$$

$$= -\frac{\partial}{\partial x_i}\left(p' v_i\right) - \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho_0 v^2\right)$$

The basic equation

The energetics of acoustic motion

- Re-write the acoustic wave equation for the potential

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\frac{p'^2}{\rho_0 c^2}+\frac{1}{2}\rho_0 v^2\right\}+\frac{\partial}{\partial x_i}\left\{p' v_i\right\}=0$$

- Definition
  - (1) Potential energy : $e_p = \frac{1}{2}\frac{p'^2}{\rho_0 c^2}$
  - (2) Kinetic energy : $e_k = \frac{1}{2}\rho_0 v^2$
  - (3) Energy flux : $\vec{I} = p'\vec{v}$

- Energy conservation

$$\frac{\partial}{\partial t}\left(e_p + e_k\right) = -\frac{\partial I_i}{\partial x_i} = -\nabla \cdot \vec{I}$$

❖ Some simple three-dimensional wave fields

- General plane wave
  - General form of waves

$$p'(\vec{x},t) = f(\vec{x}-ct) + g(\vec{x}+ct)$$

  - The pressure perturbation corresponds to waves propagating along the one direction

$$p'(\vec{x},t) = Ae^{i(\omega t - \vec{k}\cdot\vec{x})}$$

where ***k***,

$$\vec{k} = (k_1, k_2, k_3)$$

is called the wave number vector.

❖ Some simple three-dimensional wave fields

- Spherical wave
  - A wave in which all the flow parameters are functions of the radial distance, ***r***, and time, *t*, only.
  - The pressure gradient polar coordinates is given as follow.

$$\nabla^2 p' = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p'}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p'}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p'}{\partial\varphi^2}$$

  - Since p' is independent of θ and φ, the wave equation reduces to

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p'}{\partial r}\right)$$

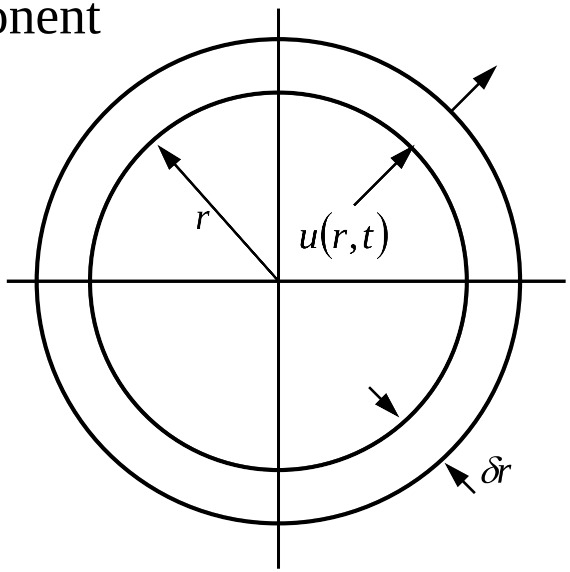
❖ Some simple three-dimensional wave fields

- Spherical wave (Alternative derivation)
  - Conservation of mass

$$4\pi r^2 \delta r \frac{\partial \rho'}{\partial t} = -\rho_0 \delta\left(4\pi r^2 u\right) \underset{\text{integral}}{\Longrightarrow} r^2 \frac{\partial \rho'}{\partial t} = -\rho_0 \frac{\partial}{\partial r}\left(ur^2\right)$$

  - Conservation of momentum of radial component

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial r}$$



  - Eliminate u and ρ' from above equations

$$\frac{r^2}{c^2}\frac{\partial^2 p'}{\partial t^2} = \frac{\partial}{\partial r}\left(r^2 \frac{\partial p'}{\partial r}\right)$$

Some simple three-dimensional wave fields

- Spherical wave
  - Consequently, a little algebra shows that this equation is precisely the same as

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rp') - \frac{1}{r^2}\frac{\partial^2}{\partial r^2}(rp') = 0$$

  - This satisfies one-dimensional wave equation :

$$rp'(r,t) = f(r-ct) + g(r+ct)$$

  - So that,

$$p'(r,t) = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r}$$

❖ Some simple three-dimensional wave fields

- Causality condition (Sommerfeld radiation condition)
  - In open space, *information about the current source activity should not be contained in past waves*. (The sound must not anticipates its cause.)
  - The solution of the wave equation in open space consists entirely of *outward travelling waves*.
  - Specifically, Sommerfeld required that the limit

$$\lim_{t\to\infty} r\left\{\frac{\partial p'}{\partial t}+c\frac{\partial p'}{\partial r}\right\}=0$$

Some simple three-dimensional wave fields

- Spherical wave
  - This is a test for their *physical reasonableness*. Only the outward wave component is admitted.

$$p'(r,t) = \frac{f(r-ct)}{r}$$

  - For example, for an outward propagating spherically symmetric wave of frequency ω,

$$p' = \frac{Ae^{i\omega(t-r/c)}}{r}$$

where A is a complex constant.

❖ Some simple three-dimensional wave fields

- Spherical wave
  - From conservation of the radial component momentum,

$$u(r,t)=\frac{A}{i\omega\rho_0}\left\{\frac{i\omega}{cr}+\frac{1}{r^2}\right\}e^{i\omega(t-r/c)}$$

$$=\frac{A}{r}e^{i\omega(t-r/c)}\frac{e^{-i\phi}\sqrt{1+(c/\omega r)^2}}{\rho_0 c}$$

$$=\frac{\sqrt{1+(c/\omega r)^2}}{\rho_0 c}\,p'\left(r,t-\frac{\phi}{\omega}\right)$$

where φ, the phase angle, is

$$\phi=\tan^{-1}(c/\omega r)$$

❖ Some simple three-dimensional wave fields

- Spherical wave (Note : special features)

1. Pressure & velocity perturbation are not in phase

2. Impedance is less then that of the plane wave

$$z \equiv \frac{p'}{u} = \frac{1}{\sqrt{1+\left(c/\omega r\right)^2}}\rho_0 c < \rho_0 c$$

3. For small r (for NEAR field)

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial r} = \frac{A}{r} e^{i\left(t-\frac{r}{c}\right)}\left\{\frac{1}{r}+\frac{i\omega}{c}\right\} \approx \frac{p'}{r} \quad \Rightarrow \quad p' \sim \frac{\partial u}{\partial t}$$

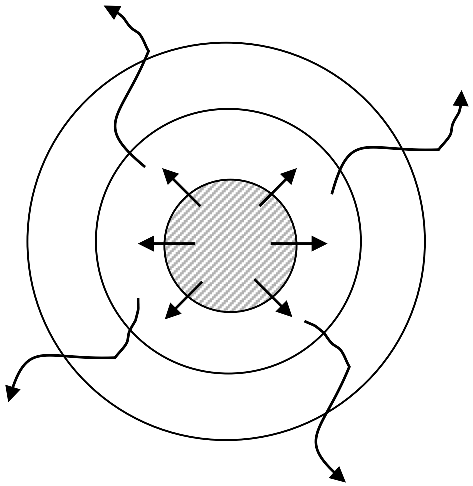
4. For large r (for FAR field)

$$\lim_{r\to\infty}\phi = \lim_{r\to\infty}\tan^{-1}\left(\frac{c}{\omega r}\right) = 0 \quad \Rightarrow \quad u(r,t) = \frac{1}{\rho_0 c} p'(r,t)$$

The pressure and velocity are nearly in phase and the spherical wave *behaves like a plane wave*.

Some examples of 3-D waves

- The sound scattered by a small bubble
  - If the fluid is in unsteady motion, the pressure in the vicinity of the bubble will vary, and the bubble will respond by pulsating and thereby *producing a secondary centered sound wave*.
  - In practice, it is occurred from metallic casting, bubble in the fluid, cavitations, and boiling coolant, etc.
  - Assumption:
    - High frequency approximation
    - The case which is the bubble pulsate with large amplitude and at very low frequency will be consider later more thoroughly.



❖ Some examples of 3-D waves

- The sound scattered by a small bubble
  - The surface of the void is unable to sustain any pressure variation the sum of the incident and scattered pressures must vanish at the boundary surface $r=a$.

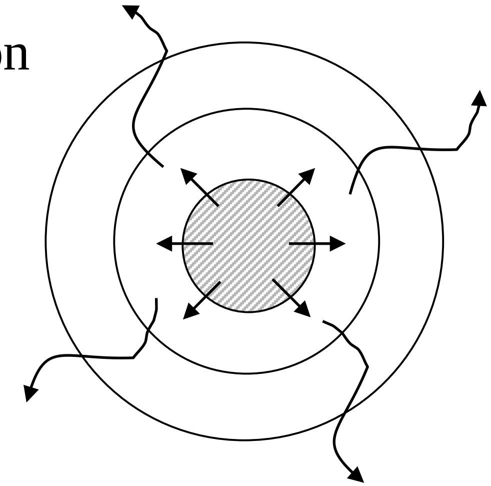
$$p'_i(t) + p'(a,t) = 0$$

where

$p'_i(t)$: environmental pressure fluctuation

$p'(r, t)$: centered outgoing sound wave

$$p'(r,t) = \frac{f(r-ct)}{r}, \qquad r \geq a$$



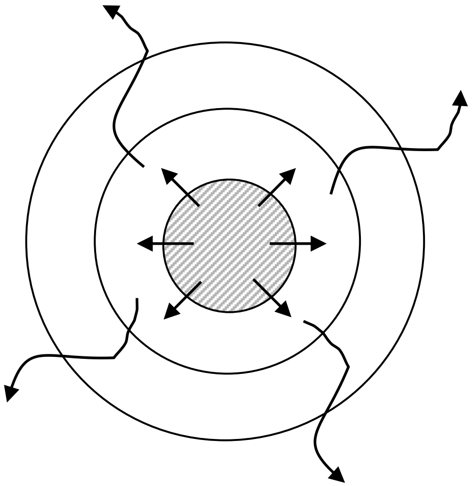
❖ Some examples of 3-D waves

- The sound scattered by a small bubble
  - From the boundary condition,

$$p'(a,t) = \frac{f(a-ct)}{a} = -p'_i(t)$$

  - Let *X*=*a*-*ct*,

$$f(X) = -a p_i'\left(\frac{a-X}{c}\right)$$

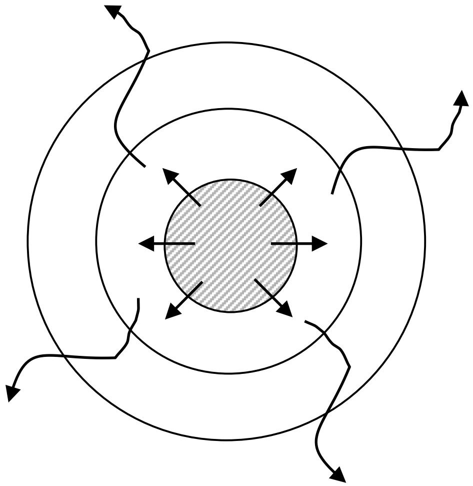


❖ Some examples of 3-D waves

- The sound scattered by a small bubble
  - Hence,

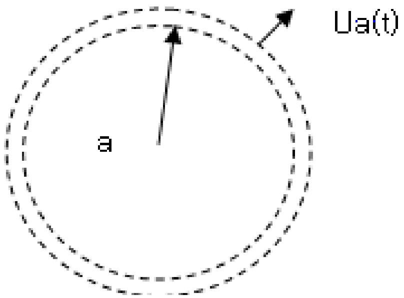
$$p'(r,t) = \frac{f(r-ct)}{r} = -\frac{a}{r} p_i'\left(t - \frac{r-a}{c}\right)$$

  - The scattered pressure wave travels radially outwards with its amplitude inversely proportional to the distance travelled.
  - The pressure time history at the void is faithfully mimicked at a time *(r-a)/c* later by the sound pressure at *r*.



Some examples of 3-D waves

- The sound generated by a pulsating sphere
  - Suppose that the radial velocity at radius a on the surface of the spherical radiator is specified to be $u_a(t)$.
  - Pressure field generated by this vibration



$$p'(r,t) = -\frac{a}{r} p'_i\left(t - \frac{r-a}{c}\right) = -\frac{a}{r} p'\left(a, t - \frac{r-a}{c}\right)$$

  - The radial component of the momentum balance is

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial r} = \frac{a}{r^2} p'\left(a, t - \frac{r-a}{c}\right) + \frac{a}{rc} \frac{\partial p'}{\partial t}\left(a, t - \frac{r-a}{c}\right)$$

❖ Some examples of 3-D waves

- The sound generated by a pulsating sphere
  - Consider a time harmonic wave with angular frequency ω. Let

$$p'(r,t)=\hat{p}(r)e^{i\omega t},\quad u(r,t)=\hat{u}(r)e^{i\omega t},\quad u_a(t)=\hat{u}_a e^{i\omega t}$$

  - Then, the momentum equation becomes

$$i\omega\rho_0\hat{u}=e^{-i\omega(r-a)/c}\left\{1+\frac{i\omega r}{c}\right\}\frac{a}{r^2}\,\hat{p}(a)$$

  - At $r=a$, $u=u_a$, so that

$$\hat{p}(a)=\frac{ai\omega\rho_0\hat{u}_a}{1+\dfrac{i\omega a}{c}}$$

Some examples of 3-D waves

- The sound generated by a pulsating sphere
  - the complete field induced by the pulsating spherical boundary is

$$\hat{p}(r)=\frac{a}{r}\frac{\frac{i\omega a}{c}}{1+\frac{i\omega a}{c}}\rho_0 c\hat{u}_a e^{i\omega(r-a)/c}$$

Some examples of 3-D waves

- The sound generated by a pulsating sphere
  - The (specific) radiation impedance, *Z*
    - The radiation impedance, *Z*

$$Z = \frac{p'}{u}$$

    - This is often presented as a fraction of a plane's acoustic impedance, $\rho_0 c$. That is termed the specific radiation impedance

$$\frac{Z}{\rho_0 c} = \frac{\dfrac{i\omega a}{c}}{1+\dfrac{i\omega a}{c}} = \frac{1+i\left(\dfrac{c}{\omega a}\right)}{1+\left(\dfrac{c}{\omega a}\right)^2}$$

    - When $\omega a/c$ is large the specific radiation impedance is purely real and equal to unity, but when $\omega a/c$ is small the specific radiation impedance is purely imaginary.

❖ Some examples of 3-D waves

- The sound generated by a pulsating sphere
  - **Helmholtz number (Compactness ratio)**

$$\frac{\omega a}{c} = \frac{2\pi a}{\lambda}$$

  - Case I : $\omega a/c \gg 1$
    - the circumstance of sphere is larger than the acoustic wave length (the sphere is large enough to be called ***non-compact***)

$$p'(r,t) = \frac{a}{r}\rho_0 c u_a\left(t - \frac{r-a}{c}\right), \qquad \frac{\omega a}{c} >> 1$$

    - The entire disturbance field set up by the pulsations of the sphere is simple and wave-like.
    - Z becomes purely real : $Z = 1$

❖ Some examples of 3-D waves

- The sound generated by a pulsating sphere
  - Case II : $\omega a/c \ll 1$
    - the circumstance of sphere is smaller than the acoustic wave length (the sphere is small enough to be called ***compact***)

$$p'(r,t) = \frac{a^2}{r}\rho_0 \frac{\partial u_a}{\partial t}\left(t - \frac{r-a}{c}\right), \qquad \frac{\omega a}{c} << 1$$

    - the vicinity of the vibrating surface is not wave-like.
    - It's virtually independent of the compressibility.
    - Z becomes purely imaginary : $Z = i\left(\frac{\omega a}{c}\right)$

Some examples of 3-D waves

- The sound generated by a pulsating sphere
  - By comparing two cases,

$$\hat{p}(r)_{compact} = \left(\frac{i\omega a}{c}\right)\hat{p}(r)_{non-compact}$$

  - This implies that
    1. Small acoustic radiator are very inefficient and their response increase with frequency.
    2. Large acoustic radiator are efficient and their response remain same over all frequency range.

❖ Some examples of 3-D waves

- Resonant scatterer
  - The sound generated by an isolated gas bubble with large amplitude and at low frequency

$$\hat{p}(a,t) = \frac{a_0 i\omega\rho_0 \hat{u}_a}{1+\dfrac{i\omega a_0}{c}} e^{i\omega t}$$

where

a_0 : mean radius

$\hat{p}, \hat{u}_a$: complex amplitudes of pressure and velocity

  - Total surface pressure on the bubble's external surface is

$$p_0 + p'_i + \hat{p}(a,t)$$

where p_0 is the mean pressure in the external fluid.

Some examples of 3-D waves

- Resonant scatterer
  - Pressure inside the bubble, p_g is

$$p_g = p_0 + p'_i + \hat{p}(a,t) + \frac{2T}{a}$$

  - Assume that the gas inside the bubble is perfect gas and in an adiabatic state, then

$$p_g a^{3\gamma} = p_g a_0^{3\gamma}$$

  - Differentiating above equation w.r.t. time,

$$\frac{\partial p_g}{\partial t} a_0^{3\gamma} + p_{g0} 3\gamma a_0^{(3\gamma-1)} \frac{\partial a}{\partial t} = \frac{\partial p_g}{\partial t} a_0^{3\gamma} + p_{g0} 3\gamma a_0^{(3\gamma-1)} u_a = 0$$

$$\frac{\partial}{\partial t} p_g = -\frac{3 p_{g0} \gamma}{a_0} u_a$$

❖ Some examples of 3-D waves

- Resonant scatterer
  - The bubble surface velocity can be determined in terms of the incident pressure field.

$$\hat{u}_a = \frac{i\omega}{\rho_0 a_0}\hat{p}_i\left\{\frac{\omega^2}{\left(1+i\omega a_0/c\right)} - \left(\frac{3\gamma p_{g0}}{\rho_0 a_0^2} - \frac{2T}{\rho_0 a_0^3}\right)\right\}^{-1}$$

$$= \frac{i\omega}{\rho_0 a_0}\hat{p}_i\left\{\frac{\omega^2}{\left(1+i\omega a_0/c\right)} - \left(\frac{3\gamma p_0}{\rho_0 a_0^2} + \left(3\gamma - 1\right)\frac{2T}{\rho_0 a_0^3}\right)\right\}^{-1}$$

$$= \frac{i\omega}{\rho_0 a_0}\hat{p}_i\left\{\frac{\omega^2}{\left(1+i\omega a_0/c\right)} - \omega_0^2\right\}^{-1}$$

❖ Some examples of 3-D waves

- Resonant scatterer
  - For air and vapor bubbles in water,

$$\frac{\omega a_0}{c} << 1$$

so that the bubble response is seen to be that of a lightly damped oscillator which resonates ate the frequency ω_0.

$$\omega_0^2 = \left(\frac{3\gamma p_0}{\rho_0 a_0^2} + (3\gamma - 1)\frac{2T}{\rho_0 a_0^3} \right)$$

  - Surface tension
    - Acts to raise the resonance frequency but not usually by very much
    - Can be negligible for bubbles resonant below 30kHz

A more elaborate 3-D wave field

- A whole variety of solution

$$\frac{\partial^2\varphi}{\partial t^2}-c^2\nabla^2\varphi=0 \quad \Rightarrow \quad \frac{\partial^2}{\partial t^2}\frac{\partial\varphi}{\partial x_i}-c^2\nabla^2\frac{\partial\varphi}{\partial x_i}=0$$

- One such pressure field can be obtained

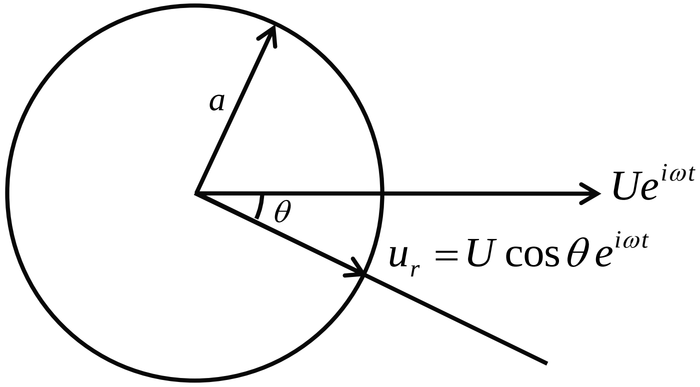
$$p'(\mathbf{x},t)=\frac{\partial}{\partial x_1}\left\{\frac{f(r-ct)}{r}\right\}=\cos\theta\frac{\partial}{\partial r}\left\{\frac{f(r-ct)}{r}\right\}$$

where θ is the angle between the 1-axis and the position vector $\boldsymbol{x}$, $\cos\theta=x_1/r$.

❖ A more elaborate 3-D wave field

- The sound of an oscillating rigid sphere
  - Consider a rigid sphere whose center moves along the x_1 axis in a small amplitude harmonic motion about the origin of coordinates.
  - The sound field induced by the sphere's vibration is actually the field of the form expressed with harmonic time dependence;

$$p'(\vec{x},t) = A\cos\theta \frac{\partial}{\partial r}\left\{\frac{e^{i\omega(t-r/c)}}{r}\right\}$$



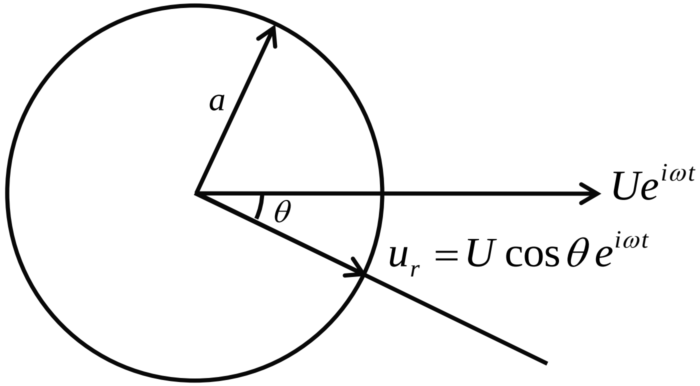
❖ A more elaborate 3-D wave field

- The sound of an oscillating rigid sphere
  - The complex amplitude *A* can be determined by applying the radial momentum equation at the surface of the sphere

$$\rho_0 \frac{\partial}{\partial t} = i\omega\rho_0 u = -\frac{\partial p'}{\partial r} = -A\cos\theta \frac{\partial^2}{\partial r^2}\left\{\frac{e^{i\omega(t-r/c)}}{r}\right\}$$

  - So that

$$u = \frac{-\cos\theta}{i\omega\rho_0}\left\{\frac{2}{r^3} + \frac{2i\omega}{cr^2} - \frac{\omega^2}{c^2 r}\right\} A e^{i\omega(t-r/c)}$$



A more elaborate 3-D wave field

- The sound of an oscillating rigid sphere
  - On the surface of sphere,

$$A = \frac{-i\omega\rho_0 U a^3 e^{i\omega a/c}}{2(1+i\omega a/c) - \omega^2 a^2/c^2}$$

$$p'(r,\theta,t) = \frac{-i\omega\rho_0 U a^3}{2(1+i\omega a/c) - \omega^2 a^2/c^2} \frac{\partial}{\partial r}\left\{\frac{e^{i\omega(t-(r-a)/c)}}{r}\right\}$$

  - There is the parameter *ωa/c*, the compactness ratio.
    1. Compact sphere
       1. Near-field
       2. Far-field
    2. Non-compact sphere

A more elaborate 3-D wave field

- The sound of an oscillating rigid sphere
  - Compact sphere ($\omega a/c \ll 1$)
    - Radiates sound very ineffectively

$$p'(r,\theta,t) = -\tfrac{1}{2} i\omega\rho_0 U a^3 \cos\theta \frac{\partial}{\partial r}\left\{\frac{e^{i\omega(t-r/c)}}{r}\right\}$$

$$= -\frac{\omega^2 a^2}{c^2}\tfrac{1}{2}\rho_0 c U \cos\theta \frac{a}{r} e^{i\omega(t-r/c)}\left(1-\frac{ic}{\omega r}\right)$$

  - Far field
    - The pressure field is in phase with the velocity field
    - Radiates to infinity carrying energy away from the sphere
  - Near field
    - The motion is hardly influenced by the speed of sound, or, therefore, by the material's compressibility

A more elaborate 3-D wave field

- The sound of an oscillating rigid sphere
  - On the surface of the compact sphere

$$p'(a,\theta,t) = \frac{1}{2} i\omega\rho_0 aU\cos\theta\, e^{i\omega t}\left\{1 + \frac{1}{2}\left(\frac{\omega a}{c}\right)^2 - \frac{i}{2}\left(\frac{\omega a}{c}\right)^3\right\} \;;\; \frac{\omega a}{c} << 1$$

  - The reaction of the material surrounding the vibrating sphere produces the drag. Its magnitude is equal to the force required to overcome the inertia of half the mass of fluid displaced by the sphere.
    - Total drag

$$-\int_0^\pi 2\pi a\sin\theta\, p'(a,\theta,t)\cos\theta\, a d\theta = -\rho_0 \frac{2}{3}\pi a^3 i\omega U e^{i\omega t}\left\{1 + \frac{1}{2}\left(\frac{\omega a}{c}\right)^2 - \frac{i}{2}\left(\frac{\omega a}{c}\right)^3\right\}$$

    - In-phase drag

$$\frac{1}{3}\pi a^2 \rho_0 c\overline{U}^2\left(\frac{\omega a}{c}\right)^4$$

❖ A more elaborate 3-D wave field

- The sound of an oscillating rigid sphere
  - Non-compact sphere ($\omega a/c \gg 1$)
    - radiates sound very effectively
    - the surface pressure is in-phase with surface velocity.

$$p'(r,\theta,t) = \rho_0 c U \cos\theta \frac{a}{r} e^{i\omega(t-(r-a)/c)}$$

  - The drag on the non-compact sphere is
    - Total drag

$$\int_0^{\pi} 2\pi a \sin\theta \, p'(a,\theta,t) \cos\theta \, a d\theta = 2\pi a^2 \rho_0 c U e^{i\omega t} \int_0^{\pi} \sin\theta \cos^2\theta d\theta$$

    - In-phase drag

$$\frac{4}{3}\pi a^2 \rho_0 c \overline{U}^2$$

Three-dimensional wave equation

❖ Two dimensional sound waves

- The pressure in the water
  - Varies due only to hydrostatic effects
  - The vertical acceleration being negligible compared to gravity

$$p = p_0 + \rho g(\xi + h - z)$$

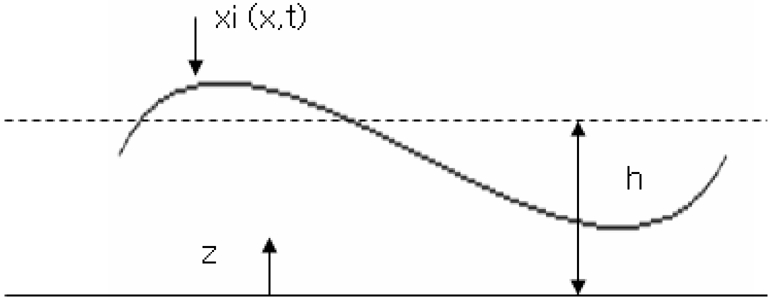
$$\rho g \frac{\partial \xi}{\partial x_\alpha} = \frac{\partial p}{\partial x_\alpha} = -\rho \frac{\partial v_\alpha}{\partial t}$$

- Mass conservation

$$\rho h \frac{\partial v_\alpha}{\partial x_\alpha} = -\rho \frac{\partial \xi}{\partial t}$$

- Combining two equations

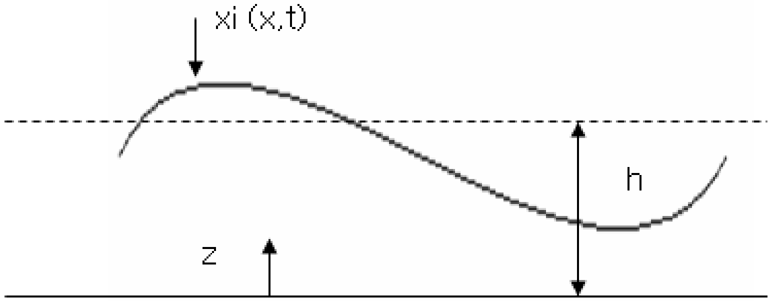
$$\frac{\partial^2 \xi}{\partial t^2} - c^2 \frac{\partial^2 \xi}{\partial x_\alpha^2} = 0$$



❖ Two dimensional sound waves

- 2-D wave equation
  - valid only when $\lambda \gg h$, an analogue of 2-D sound wave.
  - what happens if $\lambda \ll h$?
    - v_a is a function of frequency
    - groups of waves at different frequency disperse.
- General solutions
  - 1-D wave $f(x_1 - ct)$
  - 3-D wave $\frac{1}{r} f(r - ct)$
  - 2-D wave :

    Huygen's Principle does not work in Two Dimensional wave field.



Two dimensional sound waves

- A simple 2-D wave solution

$$p'(x,t) = \frac{1}{\sqrt{c^2 t^2 - r^2}}, \qquad r < ct$$
$$= 0 \qquad\qquad , \qquad r > ct$$

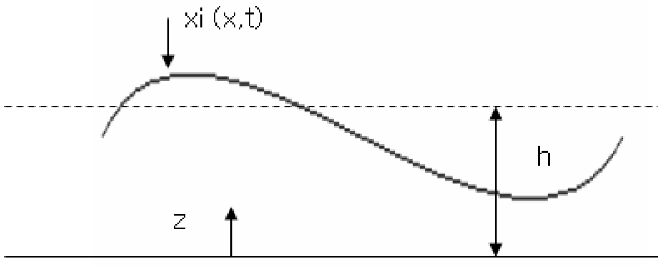
- In general,

$$p'(x,t) = \int_{-\infty}^{t-r/c} \frac{f(\tau)}{\sqrt{c^2 (t-\tau)^2 - r^2}} d\tau$$

$$c^2 (t-\tau)^2 - r^2 = \{c(t-\tau)+r\}\{c(t-\tau)-r\}$$

is singular when

$$c(t-\tau) = r$$



Two dimensional sound waves

- Approximately,

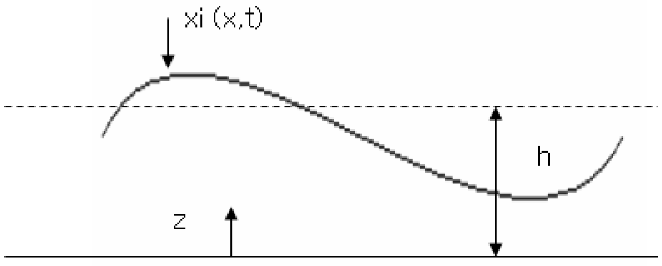
$$c^2(t-\tau)^2 - r^2 = \{c(t-\tau)+r\}\{c(t-\tau)-r\}$$
$$= 2r\{c(t-\tau)-r\}$$

- Far field form of the integral is

$$p'(x,t) \sim \frac{F(r-ct)}{\sqrt{r}}$$

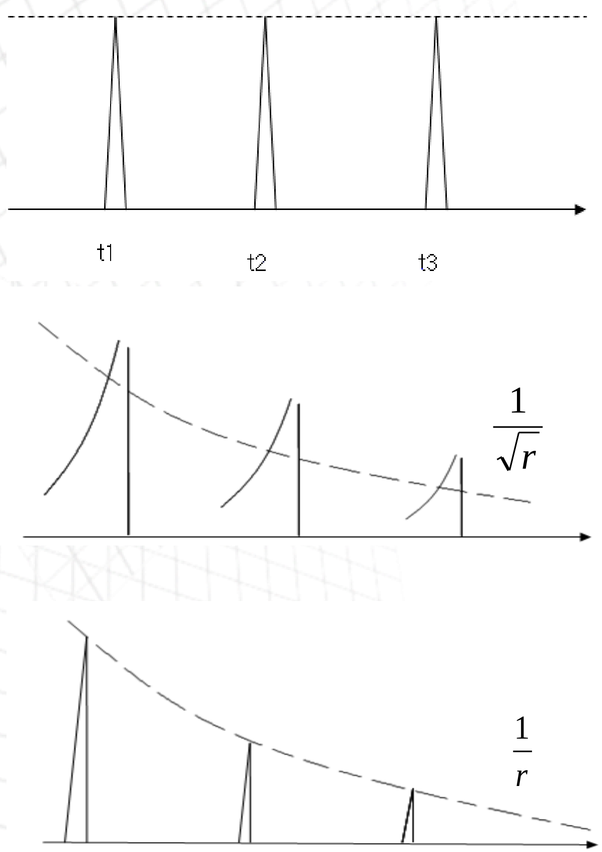
where

$$F(r-ct) = \frac{1}{\sqrt{2c}}\int_{-\infty}^{t-r/c} \frac{f(\tau)}{\sqrt{t-r/c-\tau}}\,d\tau$$



❖ Two dimensional sound waves

- Illustration of simple wave fields



- One dimensional

  : Pulse propagates with speed *c* at constant amplitude and shape.

- Two dimensional

  : Singular wave front propagates with constant speed *c*. Succeeding 'wake' becomes weaker as it travels.

- Three dimensional

  : Pulse propagates with constant shape and speed but amplitude is inversely proportional to distance travelled.

❖ Two dimensional sound waves

- Simple harmonic wave
  - Consider the sound field has frequency ω,

$$p'(r,t) = A\int_{-\infty}^{t-r/c} \frac{e^{i\omega t}}{\sqrt{c^2(t-\tau)^2 - r^2}}\, d\tau$$

  - Changing variable $s=c(t-\tau)/r$, then we have

$$p'(r,t) = \frac{Ae^{i\omega t}}{c}\int_{1}^{\infty} \frac{e^{(-i\omega sr/c)}}{(s^2-1)^{1/2}}\, ds$$

  - This integral cannot be evaluated explicitly. Instead it is given a name and tabulated in Mathematical Tables.

    → Hankel function!

❖ Two dimensional sound waves

- Simple harmonic wave
  - Hankel function: Zeroth order

$$H_0^{(2)}(X)=\frac{2i}{\pi}\int_1^{\infty}\frac{e^{-iXs}}{\left(s^2-1\right)^{1/2}}ds$$

  - Using a zeroth order Hankel function, the pressure field is simplified as

$$p'(r,t)=\frac{A\pi}{2ic}e^{i\omega t}H_0^{(2)}\left(\frac{\omega r}{c}\right)$$

Two dimensional sound waves

Simple harmonic wave

- Far field approximation

$$p'(r,t) \underset{r\to\infty}{\sim} \sqrt{\frac{\pi}{2rc\omega}} A e^{i\omega(t-r/c)-i\pi/4}$$

- Near field approximation (for small r)

$$p'(r,t) \sim -\frac{A}{c} e^{i\omega t} \log_e\left(\frac{\omega r}{c}\right)$$

- The pressure perturbation and hence the velocity potential have a logarithmic *singularity at the origin*. This is the characteristic form for the velocity potential due to two-dimensional sources in steady aerodynamics.